§4 Limits

4.1 Limits of Functions

Definition :

Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if for every $\delta > 0$, there exists at least one point $x \in A \setminus \{c\}$ such that $|x-c| < \delta$

$(\forall \delta > 0)(\exists x \in A\setminus\{c\})(|x-c| < \delta)$

Rewrite in another way :

Define $V_\delta(c) = (c-\delta, c+\delta)$

c is a cluster point of A if $(\forall \delta > 0)(V_\delta(c) \cap A\setminus\{c\} \neq \phi)$

Examples :

1) If $A = (0,1)$, then any point in $[0,1]$ is a cluster point of A.
2) If A is a finite subset of $\mathbb{R}$, then A has no cluster point.

Exercises :

1) If $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, does A have any cluster points ?
2) Find the set of cluster points of $\mathbb{Q}$.

Theorem :

$c \in \mathbb{R}$ is a cluster point of $A \subseteq \mathbb{R}$ if and only if there exists a sequence $\{a_n\}$ in $A\setminus\{c\}$ such that $\lim_{n\to\infty} a_n = c$.

proof : (Exercise)

"Definition" : (Limit of a function)

$L \in \mathbb{R}$ is said to be a limit of f at c if when x is getting closer and closer to c, but NOT equal to c, $f(x)$ is getting closer and closer to L.

If L is a limit of f at c, we denote it by $\lim_{x\to c} f(x) = L$.

Note :

1) Limit of a function at a point must be a real number.
2) We do NOT care how f behaves at the point c.

Examples:

1) Let $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x \neq 0 \end{cases}$

$\lim_{x\to 0} f(x)$ does NOT exist.

(∞ is NOT a real number, although some may write $\lim_{x\to 0} f(x) = \infty$)

2) Let $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

$\lim_{x\to 0} f(x) = 0$.

($f(0) = 1$ does NOT play any role!)

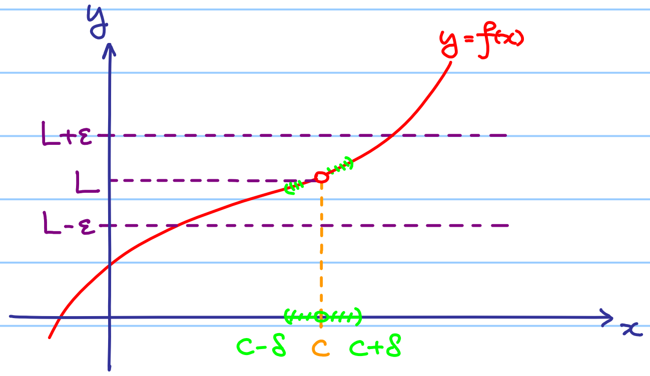
Definition:

Let $A \subseteq \mathbb{R}$ and let c be a cluster point of A. For $f: A \to \mathbb{R}$, $L \in \mathbb{R}$ is to be a limit of f at c if

for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ and $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in V_\delta(c) \cap A \setminus \{c\})(|f(x) - L| < \varepsilon)$

Geometrical meaning:



Remarks:

1) If $A = \{0, 1\}$, $f: A \to \mathbb{R}$,

then we cannot define $\lim_{x\to 0} f(x)$ as 0 is NOT a cluster point of A.

2) When we say $\lim_{x\to c} f(x)$, the domain of f is assumed to be the maximum domain that f can be defined.

Theorem: (Uniqueness of limits)

Let $f: A \to \mathbb{R}$ and let c be a cluster point of A.

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = L'$, then $L = L'$.

proof:

Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = L'$.

Given $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that

$|f(x) - L| < \frac{\varepsilon}{2}$ for all $x \in A$ with $0 < |x-c| < \delta_1$

$|f(x) - L'| < \frac{\varepsilon}{2}$ for all $x \in A$ with $0 < |x-c| < \delta_2$

Take $\delta = \min\{\delta_1, \delta_2\} > 0$.

Since c is a cluster point of A, $A \cap V_\delta(c)$ is nonempty.

Pick $x_0 \in A \cap V_\delta(c)$, we have

$$
\begin{aligned}
|L - L'| &= |L - f(x_0) + f(x_0) - L'| \\
&\leq |f(x_0) - L| + |f(x_0) - L'| \qquad (\delta \leq \delta_1, \delta_2 \Rightarrow |x_0 - c| < \delta_1, \delta_2) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

Since ε can be arbitrarily small, $L - L' = 0$ i.e. $L = L'$.

Exercises:

Let $b, c \in \mathbb{R}$. Prove that

a) $\lim_{x\to c} x = c$

b) $\lim_{x\to c} b = b$

c) $\lim_{x\to c} x^2 = c^2$

Remember how we convince ourselves (and students) that $\lim_{x\to 2} x+3 = 5$.

$x_1 = 1.9$ $\quad x_2 = 1.99$ $\quad x_3 = 1.999$ $\quad \cdots \quad x_n \to 2$

$f(x_1) = 4.9$ $\quad f(x_2) = 4.99$ $\quad f(x_3) = 4.999$ $\quad \cdots \quad f(x_n) \to 5$

OR

$\therefore \lim_{x\to 2} x+3 = 5$

$x_1 = 2.1$ $\quad x_2 = 2.01$ $\quad x_3 = 2.001$ $\quad \cdots \quad x_n \to 2$ $\quad \lim_{n\to\infty} x_n = c$

relation

$f(x_1) = 5.1$ $\quad f(x_2) = 5.01$ $\quad f(x_3) = 5.001$ $\quad \cdots \quad f(x_n) \to 5$ $\quad \lim_{n\to\infty} f(x_n) = L \longleftrightarrow \lim_{x\to c} f(x) = L$

Question: What is the relation between $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{x\to c} f(x) = L$?

Theorem : (Sequential criterion)

Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. $\lim_{x\to c} f(x) = L$ if and only if for every sequence $\{x_n\}$ in $A\setminus\{x_n\}$ that converges to c, we have $\lim_{n\to\infty} f(x_n) = L$.

proof :

"$\Rightarrow$" Given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - c| < \delta(\varepsilon)$

Suppose $\{x_n\} \subseteq A\setminus\{c\}$ that converges to c,

there exists $K(\delta(\varepsilon)) \in \mathbb{N}$ such that for $n \geq K(\delta(\varepsilon))$, $0 \underset{=}{<} |x_n - c| < \delta(\varepsilon)$

$\because x_n \neq c$

$\therefore |f(x_n) - L| < \varepsilon$ for all $n \geq K(\delta(\varepsilon))$

i.e. $\lim_{n\to\infty} f(x_n) = L$

"$\Leftarrow$" Prove by contradiction, suppose $\lim_{x\to c} f(x) \neq L$.

$(\exists\, \varepsilon_0 > 0)(\forall\, \delta > 0)(\exists\, x \in V_\delta(c) \cap A\setminus\{c\})(|f(x) - L| \geq \varepsilon_0)$

In particular, consider $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$,

there exists $x_n \in V_{\frac{1}{n}}(c) \cap A\setminus\{c\}$ such that $|f(x_n) - L| \geq \varepsilon_0$

$\therefore$ We obtain a sequence $\{x_n\}$ that converges to c

but $\{f(x_n)\}$ does NOT converge to L (Contradiction !).

Theorem : (Divergence criteria)

Let $f : A \to \mathbb{R}$ and let c be a cluster point of A.

a) $\lim_{x\to c} f(x) \neq L \Leftrightarrow$ There exists $\{x_n\} \subseteq A\setminus\{c\}$ such that $\lim_{n\to\infty} x_n = c$ but $\lim_{n\to\infty} f(x_n) \neq L$

b) $\lim_{x\to c} f(x)$ does NOT exist $\Leftrightarrow$ There exists $\{x_n\} \subseteq A\setminus\{c\}$ such that $\lim_{n\to\infty} x_n = c$

but $\lim_{n\to\infty} f(x_n)$ does NOT exist

Exercise :

Prove that

a) $\lim_{x\to 0} \sin(\frac{1}{x})$ does NOT exist.

b) $\lim_{x\to 0} f(x)$ does NOT exist where $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

4.2 Limit Theorem

Definition :

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. We say f is bounded on a neighborhood of c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M>0$ such that $|f(x)|<M$ for all $x \in A \cap V_\delta(c)$

Theorem :

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A and $\lim_{x\to c} f(x) = L$, then f is bounded on neighborhood of c.

proof :

Choose $\varepsilon = 1$,

there exists $\delta > 0$ such that

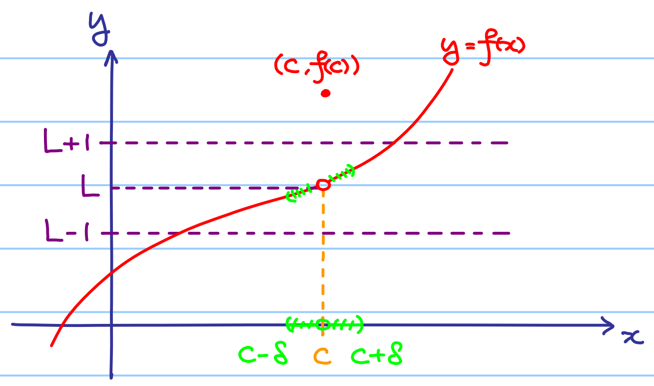
$|f(x) - L| < \varepsilon = 1 \quad \forall x \in A \setminus \{c\} \cap V_\delta(c)$

i.e. $L-1 < f(x) < L+1$

$\Rightarrow |f(x)| < |L| + 1$

Take $M = \max\{f(c), |L|+1\}$ (if $f(c)$ is defined)

or $|L|+1$ (if $f(c)$ is NOT defined)



Theorem: (Algebraic properties)

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A.

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

(1) $\lim_{x \to c} f(x) \pm g(x) = L \pm M$

(2) $\lim_{x \to c} f(x) g(x) = LM$

(3) If $g(x) \neq 0$ for all $x \in A$ and $M \neq 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

proof of (2):

Since $\lim_{x \to c} f(x) = L$, there exists $\delta' > 0$ and $Q > 0$ such that $|f(x)| < Q$ for all $x \in A \cap V_{\delta'}(c)$

Given $\varepsilon > 0$, there exists $\delta'', \delta''' > 0$ such that

$|f(x) - L| < \frac{\varepsilon}{2|M|}$ for all $x \in A \setminus \{c\} \cap V_{\delta''}(c)$ and $|g(x) - M| < \frac{\varepsilon}{2Q}$ for all $x \in A \setminus \{c\} \cap V_{\delta'''}(c)$

Take $\delta = \min\{\delta', \delta'', \delta'''\} > 0$, then for all $x \in A \setminus \{c\} \cap V_{\delta}(c)$, we have

$|f(x)g(x) - LM|$

$= |f(x)g(x) - f(x)M + f(x)M - LM|$

$\leq |f(x)||g(x) - M| + |f(x) - L||M|$

$< Q \cdot \frac{\varepsilon}{2Q} + \frac{\varepsilon}{2|M|} \cdot |M|$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

Exercise:

Prove that if $P(x)$ is a polynomial, then $\lim_{x \to c} P(x) = P(c)$.

Theorem:

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A.

If $a \leq f(x) \leq b$ for all $x \in A \setminus \{c\}$ and $\lim_{x \to c} f(x)$ exists, then $a \leq \lim_{x \to c} f(x) \leq b$.

proof: (Exercise)

Theorem: (Sandwich Theorem)

Let $A \subseteq \mathbb{R}$, $f, g, h: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A.

If $f(x) \le g(x) \le h(x)$ for all $x \in A \setminus \{c\}$ and $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} g(x) = L$.

proof:

Given $\varepsilon > 0$, there exists $\delta', \delta'' > 0$ such that

$|f(x) - L| < \varepsilon$ for all $x \in A \setminus \{c\} \cap V_{\delta'}(c)$ and $|h(x) - L| < \varepsilon$ for all $x \in A \setminus \{c\} \cap V_{\delta''}(c)$

Take $\delta = \min\{\delta', \delta''\}$, then for all $x \in A \setminus \{c\} \cap V_{\delta}(c)$, we have

$-\varepsilon < f(x) - L < g(x) - L < h(x) - L < \varepsilon$

$\therefore$ $|g(x) - L| < \varepsilon$ and so $\lim_{x \to c} g(x) = L$.

4.3 Extension of the Limit Concept

Definition:

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A.

We say that f tends to $+\infty$ $(-\infty)$ if every $M \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, we have $f(x) \ge M$ $(f(x) \le M)$.

We denote it by $\lim_{x \to c} f(x) = +\infty$ $(-\infty)$.

Remark:

Again, $+\infty$ $(-\infty)$ is just a convention, but NOT saying that limit of f at c exists!

Exercise:

Let $A = \mathbb{R} \setminus \{0\}$, $f: A \to \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$.

Show that 0 is a cluster point of A and $\lim_{x \to 0} f(x) = +\infty$.

Not surprising consequences:

Theorem:

Let $A \subseteq \mathbb{R}$, $f, g: A \to \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A.

Suppose that $f(x) \le g(x)$ for all $x \in A \setminus \{c\}$.

(a) $\lim_{x \to c} f(x) = +\infty \Rightarrow \lim_{x \to c} g(x) = +\infty$.

(b) $\lim_{x \to c} f(x) = -\infty \Rightarrow \lim_{x \to c} f(x) = -\infty$.

proof: (Exercise)

Exercises :

1) Write down the negation of the above definition .

2) Let $f: \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x} \sin \frac{1}{x}$.

   a) Show that f is unbounded .

   b) Is it true that $\lim_{x \to c} f(x) = +\infty$ or $-\infty$? Why ?

Definition :

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. Suppose that $(a, +\infty) \subseteq A$ for some $a \in \mathbb{R}$.

We say that $L \in \mathbb{R}$ is a limit of f as $x \to +\infty$ if

for all $\varepsilon > 0$, there exists $K > a$ such that for all $x \geq K$, we have $|f(x) - L| < \varepsilon$.

We denote it by $\lim_{x \to +\infty} f(x) = L$.

Exercises :

1) Write down a definition of $\lim_{x \to -\infty} f(x) = L$.

2) Prove the uniqueness and algebraic properties of the limit.